Some Properties of Chromatic, Domination, and Independence Numbers of a Graph

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Received Date: 9/Jul /2015
Accepted Date: 11/Dec/2015

Abstract

Consider G(E,V) be a finite, undirected and simple connected graph. This paper includes study of some properties that illustrate the relations between three numbers in a graph which are chromatic, domination, and independence with special restrictions. Finally, we compute these parameters in a new graph namely K4 -isosceles triangular graph.

Keywords

Chromatic number, Domination number, Independence number, and -isosceles triangular graph

1 Introduction

For a vertex v∊V(G), the open neighborhoodN(v) is the set of all vertices adjacent to v, and the closed neighborhood of v is N[v]=N(v)∪{v}.

Degree of a vertex vof any graph G is defined as the number of edges incident onv. It is denoted by deg(v) or d(v) that means d(v)=N(v). A vertex of degree 0 is an isolated vertex. The minimum and maximum degrees of vertices in G denoted by δ(G) and Δ(G), respectively. A null graph is defined as a graph without any edges. G-eis the graph obtained from a graph G by deleting the edge e of a graph G.A graph is called complete of order n (K_n) if each vertex is of degree n-1. A subgraph H of a graph G is said to be induced (or full) if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. If H is an inducedsubgraph of G with S is a set of its vertices then H is said to be induced by S and denoted by G [S]. An independent set or stableset is a set of vertices in a graph G, where no two of which are adjacent. An independence number denoted by β (G) of a graph Gis the cardinality of a maximum independent set of G.A set D ⊆V(G) is a dominating set in G if N (v)∩D≠∅; for every vertexv∈V(G)-D.The domination number of G, denoted by γ (G) , is the minimum cardinality over all dominating sets in G.

Various types of domination of a graph G have been defined and studied by several authors and more than 75 models of domination are listed in the Appendix of Haynes [6].

A vertex-coloring of G is an assignment of colors to all its vertices such that all pairs of adjacent vertices are assigned different colors. The chromatic numberχ (G) is the smallest number of colors necessary for coloring G.

In [5] A.A.Omran and E.A. El-seidy found some relations between domination numbers and the independence number in some graphs. There are many restrictions to find the relations between chromatic and domination numbers with the largest degree in a graph and also the independence number with the degree of each vertex. The following theorems illustrate this relation with special restrictions.

Theorem 1.1, [1].

For any graph G,χ(G)≤Δ(G)+1with equality if and only if either Δ(G)≠2 and G has a subgraph $K_{(\Delta(G)+1)}$ as a connected component or Δ(G)=2 and G has a cycle C_{2k+1} as a connected component.

Theorem 1.2, [1].

For any graph G with |G|=n

$\left\lceil\frac{n}{1+\Delta(G)}\right\rceil \leq \gamma(G) \leq n-\Delta(G)$.

Theorem 1.3, [3].

For any graph G,

$\beta(G) \geq \sum_{u\in V} \frac{1}{d(u)+1}$.

Theorem 1.4, [6].

For a cycle graph of order n; n ≥ 3

$\beta(G) = \gamma(G) = \left\lfloor\frac{n}{2}\right\rfloor$.

2 Main results

In this section, the new relations between chromatic, domination, and independence numbers with special restrictions are determined as follows.

Theorem 2.1.

Let G be a connected graph of order n and contain an induced complete subgraph of order $\lceil \frac{n}{2} \rceil$, then $\gamma(G) \leq \chi(G)$.

Proof.

By hypothesis Gcontains an induced complete subgraph of order $\lceil \frac{n}{2} \rceil$, so we need at least $\lceil \frac{n}{2} \rceil$ colors to guarantee that every two adjacent vertices have different colors therefore, $\chi(G) \geq \lceil \frac{n}{2} \rceil$. We know that every vertex in the induced complete subgraph dominates to all vertices in this subgraph, so we can dominate $\lceil \frac{n}{2} \rceil$ (order of induced subgraph) by only one vertex. Every vertex vdoes which does not belong to the induced complete subgraph must be adjacent to at least one vertex since G is connected graph. Therefore, there are two cases:

Case 1. If v is adjacent to a vertex u in the induced complete subgraph then udominates to at least $\lceil \frac{n}{2} \rceil +1$ vertices, thus $\gamma(G) \leq \lceil \frac{n}{2} \rceil$.

Case 2. If v is adjacent to a vertex which does not belong to the induced complete subgraph then wis dominated by v, thus $\gamma(G) \leq \lceil \frac{n}{2} \rceil$ .Therefore, in both cases, $\gamma(G) \leq \lceil \frac{n}{2} \rceil \leq \chi(G)$.

Proposition 2.2.

Let G be a connected graph of order n and contain an induced complete subgraph of order $\lceil \frac{n}{2} \rceil$ and G-$K_{\lceil \frac{n}{2} \rceil}$ is a non-null graph, then $\beta(G) \leq \chi(G)$.

Proof.

In the same manner in Theorem 2.1, we obtain $\chi(G) \geq \lceil \frac{n}{2} \rceil$ .G-$K_{\lceil \frac{n}{2} \rceil}$ is a non-null graph, so there is at least one edge in G-$K_{\lceil \frac{n}{2} \rceil}$. Thus, the maximum number of independent vertices in G-$K_{\lceil \frac{n}{2} \rceil}$ is$\lceil \frac{n}{2} \rceil$ -1 , then $\beta(G) \leq \lceil \frac{n}{2} \rceil$.

Therefore, $\beta(G) \leq \chi(G)$.

Corollary 2.3.

Let G be a connected graph of order n and contains an induced complete subgraph of order $\lceil \frac{n}{2} \rceil$and G-$K_{\lceil \frac{n}{2} \rceil}$ is non-null graph, then

$\gamma(G) \leq \beta(G) \leq \chi(G)$.

Proposition 2.4.

In a cycle graph of order n; $n \geq 3$

$\chi(G) = \beta(G) = \gamma(G)$, if and only if either n=4 or n=7.

Proof.

Let $\chi(G) = \beta(G) = \gamma(G)$,by Theorem 1.4$\beta(G)= \gamma(G)=\lceil \frac{n}{2} \rceil$, for any cycle graph of order n.Therefore, there are two cases that depend on n as follows.

Case 1: If n is odd, then $\chi(G)=3$ by [5], so $\lceil \frac{n}{2} \rceil=3$ implies n=7.

Case 2: If n is even, then $\chi(G)=2$ by [5], so $\frac{n}{2}$ implies n=4.

Thus, we get the result.

Conversely, the assertion is clear.

Theorem 2.5.

If G is a graph, then $\chi(G)=|M|$ where

$M=\{M_i$; M_i is an independent set with largest cardinal in $[G-_{j=1}^{i-1}\cup M_j]\}$.

Proof.

Suppose that|M|=k, it is clear that M_i,i=1,2,…, kmakea partition of the set of verticesV since V=∪M_i and M_i∩M_j=∅ ∀ i≠j. So, this graph is k-colorable by assigning one color to each class M_i,i=1,2,…,k, thus χ(G)≤|M|. The graph G cannot be (k-1)-colorable, since in this case we obtain two sets from the set M having the same color and this is impossible. Therefore, χ(G)=k.

Remark 2.6.

Let G be any graph of order n, then

i) If $\chi(G) = 1$, then G is null graph. Thus,$\beta(G) = n = \gamma(G)$

ii) If G is connected and $\chi(G) = 2$, then G is a bipartite graph with two sets of vertices V_1 andV_2, $|V_1|$=m and $|V_2| = n$. Then, $\gamma(G) \geq 2$ and $\beta(G)=\max\{m,n\}$.

iii) If $\chi(G) = n$, then G is complete graph. Thus, $\beta(G) = 1 = \gamma(G)$.

Proposition 2.7.

If G is a graph and H is an induced subgraph of G , then

i) $\chi(G) \geq \chi(H)$[5].

ii) $\beta(G) \geq \beta(H)$.

iii) There is no relation between $\gamma(G)$ and $\gamma(H)$.

Proof.

ii) Since every edge in H is a member in E (the set of edges in G), so the independent vertices in H are less than or equal to those in G.Thus, $\beta(G) \geq \beta(H)$.

iii) There is no relation between $\gamma(G)$ and $\gamma(H)$. To illustrate this we take the following example.

Example 2.8.

Let G be a graph shown in Fig .(2.1)

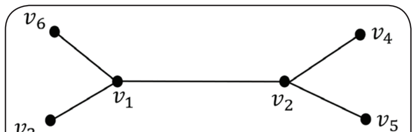

Fig .(2.1)

Now, it is obvious that $\gamma(G)=2$

Let H_1 be an induced subgraph of vertices$\{v_3,v_4,v_5,v_6\}$, then $4=\gamma(H_1) > \gamma(G)$.

Let H_2 be an induced subgraph of vertices$\{v_1,v_2\}$, then $1= \gamma(H_1) < \gamma(G)$.

Let H_3 be an induced subgraph of vertices$\{v_1,v_2,v_3,v_4,v_5\}$, then $2= \gamma(H_1) = \gamma(G)$.

Proposition 2.9.

Let G be any graph, then

i) $\gamma(G\text{-}e) \geq \gamma(G)$.

ii) $\beta(G\text{-}e) \geq \beta(G)$.

iii) $\chi(G\text{-}e) \leq \chi(G)$.

Proof.

i)Suppose G has a dominating set with the smallest cardinality D that mean that

$\gamma(G)=|D|$. If we delete any edge e from G, then we obtain a new graph G-e, so there are three cases as follows.

a) $\gamma(G\text{-}e) < \gamma(G)$,which is impossible.

b) $\gamma(G\text{-}e) = \gamma(G)$this case may occur where deleting an edge do not influence the dominating set to all vertices. For example when e join two vertices which do not belong to the dominating set.

c) $\gamma(G\text{-}e) > \gamma(G)$, again this case may occur, for example,let e=uv where v is a pendant vertex in G, such that u is dominates the vertex v in a graph

G. In G-e, the vertex v becomes isolated, so we need to add it to the set D such that D becomes the dominating set. Thus, we get the result.

i,iii) In the same manner, G-e has a new isolated vertex, so$\beta(G\text{-}e) \geq \beta(G)$ and $\chi(G\text{-}e) \leq \chi(G)$.

Proposition 2.10.

If G of order nhas k isolated vertices, then

i) $\beta(G),\gamma(G) \geq k + 1$.

ii) $\chi(G) \leq n\text{-}k$.

Proof.

i) Since there arek isolated vertices then all these vertices belong to our dominating sets and to our independent sets. Let M be a set of k isolated vertices and assuming that G[V-M] is a complete induced subgraph of G, therefore

$\beta(G[V\text{-}M]),\gamma(G[V\text{-}M])=1$, then $\beta(G),\gamma(G)=k+1$. Otherwise,

$\beta(G[V\text{-}M]),\gamma(G[V\text{-}M])>1$, then $\beta(G),\gamma(G)>k+1$. Thus, we get the result.

ii) In the same manner in (i) suppose that G[V-M] is a complete induced subgraph of G, then $\chi(G) = n\text{-}k$, we need n-k different colors, since we can color all isolated vertices by one color fromn-k different colors. Thus, we get the result.

3 K_4-isosceles triangular graph.

In this section, we will define a new graph named K_4-isosceles triangular graph is a result of augmenting n^2 of a complete graph of order 4(K_4), such that every two adjacent K_4 have one side in common, and the whole graph is an isosceles triangle with (2i-1) of K_4 graphs per row, where i=1,2,..,n. We denoted this graph by$T_{K_4}^n$.

To represent the vertices of the graph$T_{K_4}^n$ in matrix form, let r_i denote the i^{th} row measured from top to down, where i = 1,2,…, n+1. The first row r_1 which contains two vertices, the second row $r_2$which contains four vertices, and so on…, so in general the i^{th} row contains 2i vertices, except for the last row (r_{n+1}) which contains 2n vertices. Let c_j denote the j^{th} column which is numbered from the middle (the middle column has the greatest height of columns and contains two columns), where

$j = 0^{\pm},\pm1,\pm2, \ldots ,\pm(i-1)$, i = 1,2, .., n. Evidently, the middle columns $c_0\pm$contain

n+1 vertices, and hence each of the two columns c_1 and c_{-1} which lies to the right of c_0+ and to the left of c_0-, respectively, contains n vertices. In general the j^{th} column contains $n - |j|+1$ vertices. We denote the vertex of i^{th} rowand j^{th}column by $v_{i,j}$, i = 1,2,…n and $j = 0^{\pm},\pm1,\pm2,\ldots ,\pm(i - 1)$. The number of vertices in this graph is n(n+3).Fig. 3.1 shows$T_{K_4}^5$.

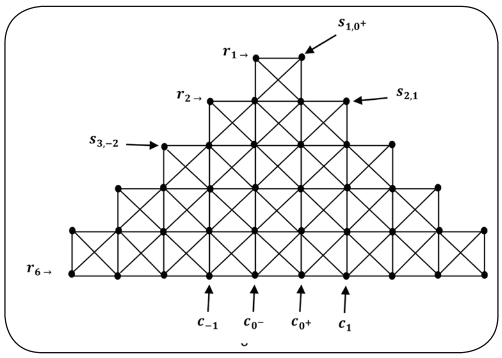

Fig 3.1: $T_{K_4}^5$.

Theorem 3.1.

Let G be an isosceles triangular graph $T_{K_4}^n$, then

i) $\chi(T_{K_4}^n) = 4$.

$$\text{ii)}\beta(T_{K_4}^n) = \begin{cases} n + \sum_{k=0}^{\lceil\frac{n-3}{2}\rceil}(1 + (n - (2k+1)), & \text{if } n \text{ is odd} \\ n - 1 + \sum_{k=0}^{\lceil\frac{n-3}{2}\rceil}(1 + (n - (2k+1)), & \text{if } n \text{ is even} \end{cases}.$$

iii) $\gamma(T_{K_4}^n) = \begin{cases} 3 + \sum_{w=0}^{\lfloor \frac{n-3}{3} \rfloor} \left(\left\lceil \frac{2n-6w}{3} \right\rceil + 2w \right), & if\ n \equiv 2 (mod\ 3) \\ 2 + \sum_{w=0}^{\lfloor \frac{n-3}{3} \rfloor} \left(\left\lceil \frac{2n-6w}{3} \right\rceil + 2w \right), & if\ n \equiv 1 (mod\ 3) \\ 1 + \sum_{w=0}^{\lfloor \frac{n-3}{3} \rfloor} \left(\left\lceil \frac{2n-6w}{3} \right\rceil + 2w \right), & if\ n \equiv 0 (mod\ 3) \end{cases}$

Proof.

i) Since G has an induced complete subgraph of order 4, then we need at least four colors (say 1,2,3,4), so $\chi(T_{K_4}^n) \geq 4$. Since each induced subgraph (K_4) has a shared two vertices (colored 1,2) with the adjacent induced subgraph (K_4), then we can color the other vertices by the remained colors (3,4) so if we colored the vertices in one induced subgraph (K_4) by colors (1,2,3,4) clockwise, then we must color the adjacent induced subgraph (K_4) counter clockwise. In the same manner, we color the remained vertices in whole graph. In this way of coloring we guarantee there are no adjacent vertices having the same color. Thus, $\chi(T_{K_4}^n)=4$.

ii) We choose the vertices from the bottom two rows $r_{n+1}$and r_n, since r_{n+1} has the greatest number of vertices which is 2n. The maximum number of vertices which can be chosen in this row such that no vertex is adjacent to other one, is n and these vertices are dominating to all vertices in row r_n. Thus, in this case we cannot choose any vertex from row r_n. Again, the maximum number of vertices can be chosen from the row r_{n-1} which contains 2n-2vertices is n-1vertices. We can choose these vertices as follows, at first we choose the terminal vertices in this row $v_{n-1,-(n-3)}$ and $v_{n-1,(n-3)}$ and starting with vertex $v_{n-1,-(n-3)}$, choose the vertices in the same row such that between any successive two vertices there is only one left vertex. Accordingly, we have seen that these vertices cannot dominate vertex $v_{n-2,(n-2)}$, therefore choose this vertex to add to an independent set. In the same manner to the second chosen, we can choose n-3 from the row r_{n-3} which contains 2n-6 vertices and one vertex from the row r_{n-4} , and so on…, (as an example, see Fig. 3.2).

There are two cases that depend on n as follows.

a) If n is odd then

$$\beta(K) = n + \sum_{k=0}^{\left\lceil \frac{n-3}{2} \right\rceil} \left(1 + (n - (2k+1)\right).$$

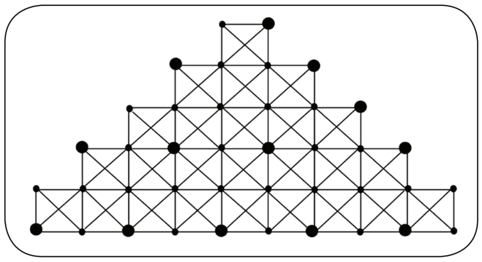

Fig 3.2 :Independence for $T_{K_4}^n$.

b) If n is even, then the last value of summation applies in row r_1, so we choose only one vertex and cannot add a vertex in row above. Thus,

$$\beta(K) = n - 1 + \sum_{k=0}^{\left\lceil \frac{n-3}{2} \right\rceil} \left(1 + (n - (2k+1)\right).$$

iii) To explain the idea of this proof, we take the three bottom rows. We can dominate these rows by choosinga number of vertices in row r_n. We apply the same idea for the following three rows. In any row, the maximum number of vertices can be dominated by one vertex which is three,by choosing this vertex in the middle of each three successive vertices. The row r_n contains 2n, so we need at least $\left\lceil \frac{2n}{3} \right\rceil$vertices to dominate this row. To dominate all the vertices of rows r_{n+1},r_n and r_{n-1} by $\left\lceil \frac{2n}{3} \right\rceil$ vertices, we choose these vertices from the row r_n. In this chosen, we must choose the two vertices: $v_{(n),-(n-2)}$ and $v_{(n),(n-2)}$, to assure dominating the two vertices $v_{n+1,-(n-1)}$ and $v_{n,(n-1)}$ (as an example, see Fig. (3.3) ; the set of big bold vertices is the dominating set for this slide of the graph with minimum cardinality).

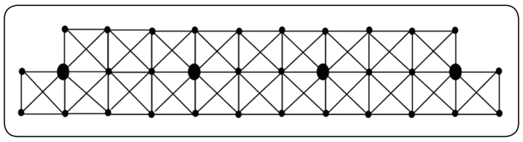

Fig (3.3):Dominating the three bottom rows of $T^6_{K_4}$.

The second slide which contains three rows r_{n-2}, r_{n-3}, and r_{n-4} is different from the first slide, since in the first slide there are two rows having the same length, but from the second slide to above the length of any row is less than from the previous row by two vertices.

In the same manner in slide one, we choose the $\left\lceil \frac{2n-6}{3} \right\rceil$verticesfrom row r_{n-3}such that we choose the vertices $v_{(n-3),-(n-5)}$and $v_{(n-3),(n-5)}$to be added to the dominating set. We choose the other vertices between them such that there are at most two vertices between any two successive vertices in the dominating set. These vertices do not dominate to the vertices $v_{(n-2),-(n-3)}$and$v_{(n-2),(n-3)}$, so we must add them to the dominating set. Therefore, in this slide we need $\left\lceil \frac{2n-6}{3} \right\rceil + 2$ vertices to dominate it. By continuingin same manner with other slides until reaching to the last slide (top slide). Thus, there are three cases that depend on the number of rows in last slide as follows.

a) If it contains three rows, then we choose the vertices $v_{3,-1}$, $v_{3,1}$, and $v_{1,0}^{+}$ to dominate the last slide. Thus, $\gamma(K) = 3 + \sum_{w=0}^{\left\lfloor \frac{n-3}{3} \right\rfloor} \left(\left\lceil \frac{2n-6w}{3} \right\rceil + 2w \right)$

b) If it contains two rows, then we choose the vertices $s_{2,0^-}$ and $s_{2,0^+}$, to dominate the last slide. Thus, $\gamma(K) = 2 + \sum_{w=0}^{\left\lfloor \frac{n-3}{3} \right\rfloor} \left(\left\lceil \frac{2n-6w}{3} \right\rceil + 2w \right)$

c) If it contains one row, then we choose the vertexv_(1,0^-) to dominate the last slide. Thus, $\gamma(K) = 1 + \sum_{w=0}^{\left\lfloor \frac{n-3}{3} \right\rfloor} \left(\left\lceil \frac{2n-6w}{3} \right\rceil + 2w \right)$.

3.2. Remark

In $T^n_{K_4}$ graph

i) $\gamma(T^n_{K_4}) < \chi(T^n_{K_4})$if and only if $n \le 2$.

ii) $\chi(T^n_{K_4}) < \gamma(T^n_{K_4}) < \beta(T^n_{K_4})$; $\forall n > 2$.

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