Applications of N_α and Generalized N_α- Closed Sets in Topological Spaces

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الخلاصة

الهدف من البحث هو تقديم بعض التطبيقات للمجموعات المغلقة من النمط N_α في الفضاءات التبولوجية مع بعض خواصها أضافة لذلك درسنا صنف جديد من المجموعات المغلقة من النمط N_α والتي تسمى المجموعات المغلقة المعممة من النمط N_α مع بعض تطبيقاتها.

الكلمات المفتاحية

الفضاءات التبولوجية، المجموعات المغلقة من النمط N_α، المجموعات المفتوحة من النمط N_α.

Abstract

the goal of this paper is to introduce some applications of N_α-closed sets in topological spaces with some of their properties. Furthermore we studied a new class of N_α-closed sets which we called generalized N_α-closed sets with some of their applications.

Keywords

Topological spaces, N_α-open set, N_α-closed set.

1. Introduction and Preliminaries

The concept of α-open (A subset A is α-open if A ⊆ int (cl(int(A))) was introduced, for the first time, by O. Njasted in 1965.The complement is called α-closed set i.e B is α-closed set if cl(int(cl(B)))⊆B. Njasted proved that the family of α-open sets in a space (X,τ) is a topology on X, which is finer than τ, and is denoted by τ_α. For more de tails see [1], [2]. The concept of pre-open (A subset A is pre-open if A⊆int(cl(A)) was introduced, for the first time by A. S. Mashhour, in 1982 ,see [3]. The concept of N_α-open set (a subset A is called N_α-open if there exists non empty α-open set B such that cl(B) ⊆ A) was first studied by N.A. Dawood, N.M. Ali in 2015. The complement is called N_α-closed set, see [4]. The class of all α-open ,pre-open, and N_α- open sets in (X,τ) are denoted by αO(X),PO(X) and $N_\alpha O(X)$ respectively and their complements are denoted by αC(X),PC(X) and N_α C(X) respectively. Thought this paper X is a topological space or a space only without assumed separation axioms unless explicitly stated. The closure and the interior of a subset A of topological space will be denoted by int(A) and cl(A) respectively.

2. Some Applications of Nα-closed Sets

Here, we shall give some new characterizations of Nα-closed sets, see the following:

2.1. Definition [4]:

A subset A of a topological space X is called Nα-closed if A^c is Nα-open set.

2.2. Theorem:

A subset A is N_α-closed iff there exists α-closed set B≠X such that A⊆int(B).

Proof: Suppose A is N_α-closed set, thus A^c is Nα-open set, hence ∃ α-open set V ≠∅such that cl(V) ⊆A^c, hence, $A=(A^c)^c\subseteq$ (cl(v)) c=int $V^c$where V^c is α-closed set ≠X. Conversely suppose there exists, α-closed set B such that B≠X and, A⊆int (B)⟹$(int(B))^c \subseteq A^c \Longrightarrow cl(B^c)\subseteq A^c$ where B^c is nonempty α-open set ⟹A^c is Nα-open set⟹A is Nα-closed set.■

2.3. Remark:

In every topological space X and ∅ are both Nα-open and Nα-closed sets. (i.e. Nα-clopen set).

2.4. Remark:

In Discrete topological space every set A is both Nα-open and Nα-closed.

Proof: Obvious

2.5. Remarks:

i. In (R, τu) every open, closed interval are Nα-closed set, where τu is usual topology on real number R.

Proof: Let A be an open interval, i.e.: A=(a,b), then∃ closed set ,so it is α-closed set say B=[a,b]such that A⊆int (B)=(a,b). On the other hand let C= [a, b] be closed interval, then ∃ closed set,so it is α-closed set D=[c,d] such that c<a<b<d such that [a,b] ⊆ int (D)=(c,d) such that c<a<b<d.■

ii. Every finite set in (R, τu) is Nα-closed set where τu is usual topology on real number R.

Proof: Let A ={x1, x2, …xn}, then for each xi ∃α-closed set say [a, b] such that xi∈ [a, b], then Ai⊆ ∪"int" $[a_i,b_i]$= $\cup(a_i,b_i)$. i = 1 ….n.■

iii. Every clopen set is Nα-closed set.

Proof: Let A≠ϕ be clopen set, thus A = int (A) (sine A is open), hence, also A is closed set so it is α-closed set, thus A is Nα-closed set.■

iv. Every open α-closed set is Nα-closed set.

Proof: Obvious.

v. Every pre-open and closed set is Nα-closed set.

Proof: Let A be pre-open set ⟹ A ⊆ int (cl(A))⟹A⊆int(A), where A is closed set, so it is α-closed set.■

2.6. Remark:

The converse of (i) in above remarks (2.5) need not be true in general, see remark(2.5(ii), where A = {x1, x2, …xn}is Nα-closed set which is not open or closed interval in (R, τu). On the other hand, the converse of (2.5(ii)) need not be true in general, see remark(2.5(i), where (a, b) is Nα-closed set, but it is not finite set in (R, τu). The converse of(2.5(iii)) need not be true in general, see remark(2.5(ii)) where A = {x1, x2, …xn}is Nα-closed set which is not clopen set in (R, τu). The converse of (2.5(iv)) need not be true in general see the following example:

Example:

Let X={a, b, c, d}, τ={{a}, {b, c}, {a, b, c}, X, ϕ}, suppose A={c}, then A is Nα closed set since there exists α-closed set B={b, c, d, e} such that A ⊆int (B)={b, c},but A is not open α-closed set.

2.7. Proposition [4]:

i. Finite union of Nα-open sets is also Nα-open set.

ii. Finite intersection of Nα-open sets is also Nα-open set.

iii. Finite union of Nα-closed sets is also Nα-closed set.

iv. Finite intersection of Nα-closed sets is also Nα-closed set.

2.8. Theorem:

If A is Nα-closed set, then so is int (A).

Proof: Since A is Nα-closed set, then there exists α-closed set, B≠X such that A⊆int(B),but int (A) ⊆A ,thus int(A) ⊆int(B),where B is α-closed set ≠X. Hence int (A) is Nα-closed set.■

The converse of theorem (2.8) need not be true in general. See the following example:

Example: Let X={a, b, c, d},τ={ X, {b}, {d}, {b, d}, ϕ}, let A={a, b, d}, then $\square_{\alpha}$C(X) ={X, {c}, {a}, {a, c, d}, {a, b, c}, {a, c}, ϕ}, we observe int (A) ={b} is Nα-closed set, but A is not Nα-closed set since ∄ nonempty α-closed set B such that A ⊆int(B).■

2.9. Lemma:

If cl(A) = X, then there is no exists α-closed set B ≠X contains A.

Proof: Suppose there exists α-closed set B≠X contains A, since B is α closed set then cl (int (cl(B))⊆ B hence cl (int (cl(A)) ⊆ clint (cl (B) ⊆ B, since cl(A) = X then, cl(int(cl(A)) = X, hence we obtain X⊆B, which is a contradiction. ■

2.10. Theorem:

If A is Nα-closed set of X,then cl(A)≠X.

Proof: Suppose, cl(A) = X, since A is Nα-closed set, then A⊆ int(B) for some set B ≠X is α-closed set on the other hand int (B) ⊆ B thus we obtain A ⊆ B, but by hypothesis cl (A) = X, thus by lemma (2.9) there is no exists α-closed set B contains A thus, we obtain a contradiction, hence cl(A)≠X.■

The converse of above theorem (2.10) need not be true in general. See the following example.

Example : Let X= {a,b,c,d,},τ={X,{b},{d},{b.d},ϕ}, C(X)={{a, c, d},{a, b, c,}, {a, c}, X, ϕ}, αC(X) = {{c}, {a}, {a, c, d}, {a, b, c}, X, ϕ}, let A= {a, c, d}, cl (A)={a, c, d}≠ X, but A is not Nα-Closed set since there is not α-closed set B≠X such that A ⊆ int(B).

Now we can make the converse of above (theorem (2.10)) is true if we add the following condition.See the following theorem:

2.11. Theorem:

If cl(A)≠X, and A is open set, then A is Nα-closed set.

Proof: We have A ⊆ cl (A) ≠X, since A is open set, thus int (A) = A⊆int (cl(A)), where cl (A) is closed set so it is α-closed set (since every closed set is α-closed set). Thus A is Nα-closed set.■

2.12. Corollary:

Let A be an open set. Then A is Nα-closed set if and only if cl(A) ≠ X

Proof: Directly by theorem (2.10) and theorem (2.11).■

2.13. Lemma [2]:

Let X1, X2 be topological spaces. Then Al and A2 are α-closed sets respectively iff Al ×A2 is α-closed set in X1×X2.

Now we shall prove the following theorem.

2.14. Theorem:

Let X1, X2 be topological spaces. Then Al and A2 are Nα-closed sets in X1, X2 respectively if and only if A1×A2 is Nα-closed set in X1×X2.

Proof: Let A1 and A2 are Nα-closed sets in X1, X2 respectively, then there exist α-closed sets B1, B2 in X1, X2 respectively such that Al ⊆ int(B1), A2 ⊆ int (B2), thus, A1 × A2 ⊆ int(B1) × int (B2)= int (B1×B2), where B1 × B2 is α-closed set in X1 × X2, see lemma (2.13). Thus, Al×A2 is Nα-closed set in X1×X2. Conversely if A1×A2 is Nα-closed set in X1×X2, then there exists basic α-closed set B1×B2 such that Al×A2⊆int (B1×B2) = int (BI)× int (B2), hence Al ⊆ int (BI), A2 ⊆ int (B2) where B1,B2 are α-closed sets in X1,X2,see lemma (2.13). Thus, Al and A2 are Nα-closed sets in X1, X2 respectively.■

2.15. Definition [4]:

Let X be a topological space, A⊆X. The Nα-closure of A is defined as the intersection of all Nα-closed sets in X containing A and is denoted by Nαcl (A).

2.16. Proposition [4]:

Let X be a topological space , A ⊆ B ⊆ X. Then:

i. Nαcl (A) ⊆ Nαcl (B)

ii. If A is Nα-closed set then A= Nαcl (A) [if X is finite set, then A is Nα closed set iff A =Nαcl (A)].

iii. x ∈Nαcl (A) iff U_x ∩ A≠∅, for each Nα-open set U containing x.

In this paper, we shall add some properties of Nα-closure (A). See the following proposition.

2.17. Proposition:

Let X be a topological space. Then:

i. Nαcl (A∪B) = Nαcl (A) ∪Nαcl (B)

ii. Nαcl(A) = Nαcl (Nαcl (A))

Proof: The proof of (i) follows by using (prop. (2.16) (i, iii)).Now we shall prove (ii) only.

Proof of (ii): We have Nαcl (A) ⊆Nαcl (Nαcl (A)). Now, to prove Nαcl (Nαcl (A) ⊆ Nαcl (A). Let x∈ Nαcl (Nαcl (A), this implies G_x∩Nαcl (A) ≠∅ for each N_α-open set G_x, thus x ∈H_x∩ A ≠∅ for each N_α-open set Hx, we get G_x∩ H_x∩ A ≠∅, but G_x∩ H_x is N_α-open set contains x. Put G_x∩ H_x = W_x, we get W_x∩ A≠∅, this implies x ∈ Nαcl (A). Thus Nαcl (Nαcl (A), = Nαcl (A).■

2.18. Lemma [5]:

Let Y be a subspace of a topological space X, such that A ⊆Y⊆ X.Then:

i. $cl_Y(A) = cl_X(A) \cap Y$

i. $int_X (A) = int_Y (A) \cap Y$

i. $int_X (A) \subseteq int_Y (A)$

2.19. Lemma [2]:

Let Y be a subspace of a topological spaceXsuch that A ⊆ Y ⊆ X.Then:

i. If A ∈α C (X),then A ∈α C (Y).

ii. If A ∈α C (Y) and Y ∈α C (X), then A ∈α C (X).

2.20. Proposition:

Let Y be a subspace of a topological space X such that A ⊆ Y ⊆ X. Then:

i. If A∈ N_α C (X),then A ∈ Nα C (Y).

ii. If A ∈ N_α C (Y) and Y is α C (X),then A ∈ Nα C (X).

Proof of (i): Let A ∈ N_αC (X), thus there exists α-closed set B ≠ X such that A ⊆ intX (B), but intX(B)⊆ int_Y (B), hence, we get A ⊆ intY (B) see (lemma (2.18) (iii)). On the other hand B is α-closed set in Y see (lemma (2.19) (i)).Thus A ∈ N_αC (Y).■

Proof of (ii): Let A ∈ N_α C (Y) where Y is α-closed set in X since A∈N_αC (Y), then there exists α-closed set B ≠ Y such that A ⊆int_Y(B) we get A∩Y⊆ int_Y(B) ∩Y, thus, we get A⊆ int_X (B) (see lemma (2.18) (ii)).On the other hand B is α-closed set in X (see lemma (2.19) (ii)).Hence A ∈ N_αC (X).■

3. Generalized Nα-closed Sets:

In this section we shall study a class of N_α-closed sets which is called generalized N_α-closed sets, with some of their properties.

3.1. Definition:

A subset A of a topological spaceX is called a generalized Nα-closed set (brieflyg $_{N\alpha}$ -closed) if Nαcl (A) ⊆ B whenever A ⊆B and B is N_α-open set.

3.2. Theorem:

X and ϕ are $g_{N\alpha}$ –closed sets of a topological space X.

3.3. Theorem:

Every Nα-closed set in X is $g_{N\alpha}$–closed set.

Proof: Let A be Nα-closed set. Let U be N_α-open set such that A ⊆ U, since A is N_α-closed set then A =Nα cl (A). Therefore Nαcl (A) U. Hence A is $g_{N\alpha}$ – closed set in X.

The converse of above theorem need not be true in general. See the following example:

Example: Let X= {a,b,c,d,},τ={X,{b},{d,{b.d},ϕ}, NαO (X) ={ X{a, c, d},{a, b, c,}, X}, NαC (X)={ϕ,{b},{d},X},let A={a, b, d},then A is$g_{N\alpha}$ –closed set in X,which is not Nα-closed set since ∄ nonempty α-closed set B≠X such that A⊆int(B).

3.4. Theorem:

The finite union of $g_{N\alpha}$ –closed sets is$g_{N\alpha}$–closed set .

Proof: Let A and B be $g_{N\alpha}$ –closed sets in X. Let G be Nα-open set in X such that A ∪ B ⊆ G then A ⊆ G and B ⊆ G. Since A and B are $g_{N\alpha}$ –closed set, thus Nαcl (A) ⊆ G and Nαcl (B) ⊆ G. Hence Nαcl (A) ∪ Nαcl (B) = Nαcl (A ∪ B) ⊆ G see ((propo.2.17(i).Therefore A ∪ B is $g_{N\alpha}$ –closed set.■

3.5. Theorem:

The finite intersection of $g_{N\alpha}$ –closed sets is $g_{N\alpha}$ –closed set .

Proof: Let A and B be $g_{N\alpha}$ –closed sets. Let G be Nα-open set in X such that A ∩ B ⊆ G, then A ⊆ G and B ⊆ G, since A and B are g_Nα^ –closed set, thus Nαcl_(A) ⊆ G and Nαcl_ (B) ⊆ G. Hence Nαcl (A ∩ B) ⊆Nαcl (A) ∩ Nαcl (B)⊆ G. Hence A ∩ B is $g_{N\alpha}$ –closed set.■

3.6. Theorem:

Let A ⊆ B ⊆ Nαcl (A), and A is $g_{N\alpha}$ –closed set, then B is also.

Proof: Let B ⊆ G, where G is Nα-open set, to prove Nαcl (B) ⊆ G,since B ⊆ G, then A ⊆ G, since A is $g_{N\alpha}$ –closed set, thus Nαcl (A) ⊆ G, since A ⊆ B ⊆ Nαcl (A), thus by using (propo. (2.16) (i)) and (propo. (2.17) (ii)), we get Nαcl (A) ⊆ Nαcl (B) ⊆ Nα cl (Nαcl (A)) = Nαcl (A), thus Nαcl (A) = Nαcl (B), hence Nαcl (B) ⊆ G, thus B is $g_{N\alpha}$ –closed set.■

3.7. Lemma [4]:

Let Y be a subspace of a topological space Xsuch that A ⊆ Y ⊆ X. Then:

i. If A∈N_αO (X), then A ∈N_αO (Y).
ii. If A ∈N_αO(Y) and Y is clopen set in X, then A ∈N_αO (X).

3.8. Theorem:

Let Y be a subspace of a topological space X such that A ⊆ Y ⊆ X.If A is g_Nα^ –closed set in X, then A is $g^{\wedge}_{N\alpha}$ –closed in Y, where Y is clopen set in X.

Proof: Let A be $g^{\wedge}_{N\alpha}$ –closed in X to prove A is $g^{\wedge}_{N\alpha}$ –closed in Y.Let A ⊆ G s.t G is Nα-open set in Y, to prove Nαcl Y(A) ⊆ G.Since G is N_α-open in Y, thus G is N_α-open set in X. Since A is $g^{\wedge}_{N\alpha}$ –closed in X,thus

Nαcl $_X$ (A) ⊆ G, thus Nαcl $_X$ (A) ∩ Y ⊆ G ∩ Y = G, hence Nαcl Y (A) ⊆ G ⇒ A is $g^{\wedge}_{N\alpha}$ –closed set in Y.■

3.9. Theorem:

In a topological space X for each x ∈ X.Then{x} is N_α-closed set or its complement X – {x} is $g^{\wedge}_{N\alpha}$ –closed set.

Proof: Suppose {x} is not Nα-closed, then X – {x} is not Nα-open and the only Nα-open set containing X – {x} is X, thus Nαcl X – {x}⊆ X, hence X – {x} is $g^{\wedge}_{N\alpha}$ –closed set.■

3.10. Theorem:

A set A is $g^{\wedge}_{N\alpha}$ –closed then Nαcl (A) – A contains no non-empty N_α-closed set in X.

Proof: We prove the result by a contradiction. Suppose there existsNα-closed set F ≠∅ such that F⊆ Nαcl (A) – A = Nαcl (A)∩ A^c. Therefore F ⊆ Nαcl (A) and F ⊆ A^c, thus A⊆F^c, since A is $g^{\wedge}_{N\alpha}$ –closed set, then Nαcl_ (A) ⊆ F^c, thus F ⊆ $(N\alpha cl\ (A))^c$. Hence F ⊆ Nαcl (A)∩$(N\alpha cl\ (A))^c$ = ∅.Thus F= ∅.Hence, Nα cl(A)-A does not containany nonempty N_α-closed set F.■

3.11. Corollary:

Let A be $g^{\wedge}_{N\alpha}$ –closed set in X where X is a finite space. A is N_α-closed if and only if Nαcl (A) – A is N_α-closed set.

Poof: If A is N_α-closed set⟹ A = Nαcl (A)⟹ Nαcl (A) – A = ∅ which is N_α-closed set. Conversely, let Nαcl (A) – A be N_α-closed set, then by (theorem (3.10)), Nαc (A) – A is does not contain any nonempty Nα-closed set, and since Nαcl (A) – A is N_α-closed subset of itself, ⟹ Nαcl (A) – A = ∅⟹ Nαcl (A) = A. Since X is finite space, then A is Nα-closed set. See (proposition (2.16)(ii)).■

3.12. Lemma [4]:

i. Let X_1, X_2 be topological spaces. Then A_1 and A_2 are N_α-open sets in X_1,X_2 if and only if $A_1 \times A_2$ is Nα_open set in $X_1 \times X_2$.

ii. Nαcl $(A_1 \times A_2)$ = Nαcl $(A_1) \times N\alpha_{cl}$ (A_2).

3.13.Theorem:

Let X_1, and X_2 are topological spaces. Then $A_1 \times A_2$is $g^{\wedge}_{N\alpha}$ –closed set in $X_1 \times X_2$ if and only if A_1 is $g^{\wedge}_{N\alpha}$ –closed in X_1, A_2 is $g^{\wedge}_{N\alpha}$ –closedin X_2.

Proof: Let $A_1 \times A_2$ be a $g^{\wedge}_{N\alpha}$ –closed set in $X_1 \times X_2$. Suppose $A_1 \subseteq B_1$and $A_2 \subseteq B_2$, where B_1, B_2 are N_α-open sets in X_1,X_2 respectively, we get $A_1 \times A_2 \subseteq B_1 \times B_2$ where $B_1 \times B_2$ is N_α-open sets in $X_1 \times X_2$, (see lemma (3.12)), thus Nαcl $(A_1 \times A_2) \subseteq B_1 \times B_2$,(sine $A_1 \times A_2$ is $g^{\wedge}_{N\alpha}$ –closed set in $X_1 \times X_2$) ⟹ Nαcl (A_1) × Nαcl_ $(A_2) \subseteq B_1 \times B_2$ ⟹ Nαcl $(A_1) \subseteq B_1$and Nαcl _ $(A_2) \subseteq B_2$.Thus A_1, A_2 are $g^{\wedge}_{N\alpha}$ –closed sets in X_1, X_2 respectively. Conversely. Let A_1, A_2 are $g^{\square}_{N\alpha}$ –closed sets in X_1, $X_2$to prove $A_1 \times A_2$ is $g^{\square}_{N\alpha}$ –closed set in $X_1 \times X_2$. Let $A_1 \times A_2 \subseteq$ B where B is N_α-open set in $X_1 \times X_2$. Put B = $U_1 \times U_2$, where U_1, U_2 are N_α-open sets in X_1,X_2respectively, thus we obtain $A_1 \subseteq U_1$and $A_2 \subseteq U_2$, since A_1, A_2 are $g^{\square}_{N\alpha}$ – closed sets in X_1, X_2 respectively, then Nαcl $(A_1) \subseteq U_1$, Nαcl $(A_2) \subseteq$U2 hence, Nαcl $(A_1) \times (A_2)$ = Nαcl $(A_1 \times A_2) \subseteq U_1 \times U_2$ (see lemma (3.12) (ii)). Thus$A_1 \times A_2$ is $g^{\square}_{N\alpha}$ –closed set in $X_1 \times X_2$. ■

4. Future Work:

We can use the concept of Nα-closed sets to study a new kind of Nα-closed mappings.

References

[1] O. Njastaf "On Some Class of Nearly Open Sets", Pacific J. Math., 15, (3), PP. 961-970, (1965).

[2] N. M. Al-Tabatabai "On Some Types of Weakly Open sets", M. Sc. Thesis. University of Baghdad, (2004).

[3] A. S. Mashhour "On Pretopological Space", Bull. Math. Soc. Sci. R. S. R. 28, No 76, PP: 39-45, (1982).

[4] N. A. Dawood, N. M. Al-Tabatabai, "Nα-Open sets and Nα-Regularity in Topological Space", International. J. of Advanced Scientific and Technical Research, 5, (3), PP. 87-95, (2015).

[5] L. J. N. Sharma, "Topology", Krishma Prakashan Medis (p)1td.India Twenty Fifth Edition, (2000).